The static quark potential to three loops in perturbation theory

Alexander V. Smirnov[a], Vladimir A. Smirnov[b], Matthias Steinhauser[c*]

[a]Scientific Research Computing Center of Moscow State University, Russia

[b]Nuclear Physics Institute of Moscow State University, Russia

[c]Institut für Theoretische Teilchenphysik, Karlsruhe Institute of Technology, Germany

The static potential constitutes a fundamental quantity of Quantum Chromodynamics. It has recently been evaluated to three-loop accuracy. In this contribution we provide details on the calculation and present results for the 14 master integrals which contain a massless one-loop insertion.

1. Introduction

The static potential enters a variety of observables connected to heavy-quark physics. Among them are the prominant examples like the determination of the bottom quark mass from Υ sum rules or the cross section for the top quark pair production close to theshold. It is desirable for both quantities to perform a third-order analysis which requires the evaluation of the static quark potential to three loops.

In order to fix the notation we write the static potential in momentum space in the following form

$$V(|\vec{q}\,|) = -\frac{4\pi C_F \alpha_s(|\vec{q}\,|)}{\vec{q}^{\,2}}\Bigg[1 + \frac{\alpha_s(|\vec{q}\,|)}{4\pi} a_1 + \left(\frac{\alpha_s(|\vec{q}\,|)}{4\pi}\right)^2 a_2 + \left(\frac{\alpha_s(|\vec{q}\,|)}{4\pi}\right)^3 \left(a_3 + 8\pi^2 C_A^3 \ln\frac{\mu^2}{\vec{q}^{\,2}}\right) + \cdots\Bigg]. \quad (1)$$

where $C_A = N_c$ and $C_F = (N_c^2-1)/(2N_c)$. In Eq. (1) we identify the renormalization scale μ^2 and the momentum transfer $\vec{q}^{\,2}$. The complete dependence on μ can easily be restored with the help of Eq. (2) of Ref. [1].

The one- and two-loop coefficients a_1 [2,3] and a_2 [4,5,6,7] are given in Eq. (4) of Ref. [1] where also the higher order terms in ε, necessary for the three-loop calculation, are presented. In 2008 the fermionic contribution [1] to a_3 was computed and end of 2009 a_3 was completed by evaluating also the purely gluonic part which was achieved by two independent groups [8,9]. In this contribution we provide some details of Ref. [8] and in particular present explicit results for the 14 master integrals containing a massless one-loop subdiagram. The results for 16 more complicated integrals have been presented in Ref. [10]. In addition one has one more finite master integral which is only known numerically and ten integrals which have no static line and are thus known since long.

*Talk presented at Loops and Legs in Quantum Field Theory 2010, Wörlitz, Germany, April 25-30, 2010

2. Reduction to master integrals

In order to compute the static potential one has to consider the four-point amplitudes describing the quark anti-quark interation. After integrating out the heavy quark mass one arrives at non-relativistic QCD and remains with only the dependence on one kinematical variable, the momentum transfer between the quarks. Consequently all occuring integrals can be mapped to one of the three integrals displayed in Fig. 1. We have performed both the direct reduction of these integrals but also applied a partial fractioning in all cases where three static lines meet in one vertex. This reduces the number of indices which have to be considered during the reduction from 15 to twelve.

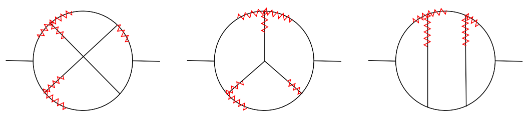

Figure 1. Three-loop two-point integrals with massless relativistic (solid lines) and static propagators (zig-zag lines).

In Ref. [8] the evaluation of a_3 has been performed for general gauge parameter ξ which is a very important check, in particular for such involved calculations as the present one. However, the computational price one has to pay is quite high: a rough estimate of the complexity based on the number of integrals which have to be reduced to masters shows that the linear ξ term is about seven times and the ξ^3 term even 18 times more complicated than the Feynman gauge result. Let us mention that nevertheless all occurring integrals could be reduced with the help of `FIRE` [11].

3. Results for master integrals

There are altogether 41 master integrals contributing to a_3. As already mentioned above, in this contribution we want to consider those with a massless one-loop insertion which can easily be integrated in terms of Γ functions using standard formulae. As a result one obtains the two-loop integrals in Fig. 2 where one of the indices has a non-integer exponent involving the space-time parameter $\varepsilon = (4-d)/2$ which is explicitly indicated next to the corresponding line. We present the analytical results for all integrals which can be expressed as one of the following functions

$$G^{(1)}_{a_1,\ldots,a_7} = \int\int \frac{\mathrm{d}k\;\mathrm{d}l}{(-k^2)^{a_1}(-(k+q)^2)^{a_2}(-l^2)^{a_3}} \times \frac{1}{(-(l+q)^2)^{a_4}(-(k-l)^2)^{a_5}(-v\cdot k)^{a_6}(-v\cdot l)^{a_7}}\,,$$

$$G^{(2)}_{a_1,\ldots,a_7} = \int\int \frac{\mathrm{d}k\;\mathrm{d}l}{(-k^2)^{a_1}(-(k+q)^2)^{a_2}(-l^2)^{a_3}} \times \frac{(-v\cdot(k-l))^{-a_7}}{(-(l+q)^2)^{a_4}(-(k-l)^2)^{a_5+\varepsilon}(-v\cdot k)^{a_6}}\,,$$

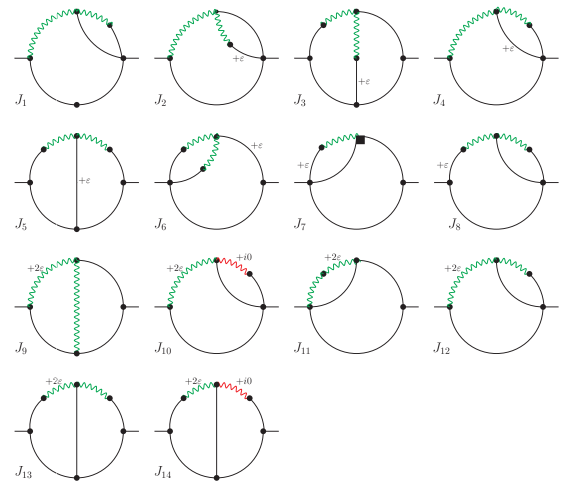

Figure 2. Two-loop master integrals with a regularized line contributing to a_3. The solid and zig-zag lines correspond to relativistic and static propagators, respectively. The black box in the case of J_7 denotes a monomial in the numerator.

$$G^{(3)}_{a_1,\ldots,a_7} = \int\int \frac{\mathrm{d}k\;\mathrm{d}l}{(-k^2)^{a_1}(-(k+q)^2)^{a_2}(-l^2)^{a_3}} \times \frac{(-v\cdot l)^{-a_7}}{(-(l+q)^2)^{a_4}(-(k-l)^2)^{a_5+\varepsilon}(-v\cdot k)^{a_6}}\,,$$

$$G^{(4)}_{a_1,\ldots,a_7} = \int\int \frac{\mathrm{d}k\;\mathrm{d}l}{(-k^2)^{a_1}(-(k+q)^2)^{a_2}(-l^2)^{a_3+\varepsilon}} \times \frac{(-v\cdot(k-l))^{-a_7}}{(-(l+q)^2)^{a_4}(-(k-l)^2)^{a_5}(-v\cdot k)^{a_6}}\,,$$

$$G^{(5)}_{a_1,\ldots,a_7} = \int\int \frac{\mathrm{d}k\;\mathrm{d}l}{(-k^2)^{a_1+\varepsilon}(-(k+q)^2)^{a_2}(-l^2)^{a_3}} \times \frac{1}{(-(l+q)^2)^{a_4}(-(k-l)^2)^{a_5}(-v\cdot k)^{a_6}(-v\cdot l)^{a_7}}\,,$$

$$G^{(6)}_{a_1,\ldots,a_7} = \int\int \frac{\mathrm{d}k\;\mathrm{d}l}{(-k^2)^{a_1}(-(k+q)^2)^{a_2}(-l^2)^{a_3}} \times \frac{(-v\cdot(k-l))^{-a_7}}{(-(l+q)^2)^{a_4}(-(k-l)^2)^{a_5}(-v\cdot k)^{a_6+2\varepsilon}}\,,$$

$$G^{(7,\pm)}_{a_1,\ldots,a_7} = \int\int \frac{\mathrm{d}k\;\mathrm{d}l}{(-k^2)^{a_1}(-(k+q)^2)^{a_2}(-l^2)^{a_3}} \times \frac{(-v\cdot l \mp i0)^{-a_7}}{(-(l+q)^2)^{a_4}(-(k-l)^2)^{a_5}(-v\cdot k)^{a_6+2\varepsilon}}\,,$$

where in all propagators, apart the last one, the causal $-i0$ is implied.

In the following we set for simplicity $q^2 = -1$ and $v^2 = 1$ since the corresponding dependences can easily be reconstructed. The results for J_1 to J_{14} read

$$J_1 = G^{(1)}_{0,1,1,0,1,1,1} = \pi^2 \Bigg[\frac{2}{3\varepsilon} + 4 + \left(24 - \frac{7\pi^2}{9}\right)\varepsilon + \left(144 - \frac{14\pi^2}{3} - \frac{352\zeta(3)}{9}\right)\varepsilon^2 + \Big(864 - 28\pi^2 - \frac{101\pi^4}{60} - \frac{704\zeta(3)}{3}\Big)\varepsilon^3 + \dots \Bigg],$$

$$J_2 = G^{(2)}_{0,1,1,0,1,1,1} = \frac{4\pi^2}{9\varepsilon} + \frac{4\zeta(3)}{3} + \frac{32\pi^2}{9} + \left(\frac{256\pi^2}{9} - \frac{34\pi^4}{45} + \frac{32\zeta(3)}{3}\right)\varepsilon + \left(\frac{2048\pi^2}{9} - \frac{272\pi^4}{45} + \frac{256\zeta(3)}{3} - \frac{1622\pi^2\zeta(3)}{27} - \frac{20\zeta(5)}{3}\right)\varepsilon^2 + \left(\frac{16384\pi^2}{9} - \frac{2176\pi^4}{45} - \frac{2339\pi^6}{630} + \frac{2048\zeta(3)}{3} - \frac{12976\pi^2\zeta(3)}{27} - \frac{968\zeta(3)^2}{9} - \frac{160\zeta(5)}{3}\right)\varepsilon^3 + \dots,$$

$$J_3 = G^{(2)}_{1,1,1,1,1,1,1} = -\frac{8\pi^2}{9\varepsilon} - \frac{20\zeta(3)}{3} - \frac{16\pi^2}{9} + \left(\frac{256\pi^2}{9} - \frac{106\pi^4}{45} - \frac{112\zeta(3)}{3}\right)\varepsilon + \left(-\frac{1792\pi^2}{9} + \frac{688\pi^4}{45} - 18\pi^4\log(2) + \frac{1216\zeta(3)}{3} + \frac{5530\pi^2\zeta(3)}{27} - \frac{1865\zeta(5)}{3}\right)\varepsilon^2 + \left(-864 s_6 + \frac{2660\zeta(5)}{3} + \frac{12778\zeta(3)^2}{9} - \frac{2440\pi^2\zeta(3)}{27} - \frac{7936\zeta(3)}{3} - 576\pi^2 \mathrm{Li}_4\left(\frac{1}{2}\right) - 24\pi^2\log^4(2) - 30\pi^4\log^2(2) + 72\pi^4\log(2) + \frac{392\pi^6}{135} - \frac{3808\pi^4}{45} + \frac{10240\pi^2}{9}\right)\varepsilon^3 + \dots,$$

$$J_4 = G^{(3)}_{0,1,1,0,1,1,1} = \frac{4\pi^2}{9\varepsilon} - \frac{8\zeta(3)}{3} + \frac{32\pi^2}{9} + \left(\frac{256\pi^2}{9} - \frac{14\pi^4}{15} - \frac{64\zeta(3)}{3}\right)\varepsilon + \left(\frac{2048\pi^2}{9} - \frac{112\pi^4}{15} - \frac{512\zeta(3)}{3} - \frac{1604\pi^2\zeta(3)}{27} + \frac{40\zeta(5)}{3}\right)\varepsilon^2 + \left(\frac{16384\pi^2}{9} - \frac{896\pi^4}{15} - \frac{6161\pi^6}{1890} - \frac{4096\zeta(3)}{3} - \frac{12832\pi^2\zeta(3)}{27} + \frac{1936\zeta(3)^2}{9} + \frac{320\zeta(5)}{3}\right)\varepsilon^3 + \dots,$$

$$J_5 = G^{(3)}_{1,1,1,1,1,1,1} = -\frac{8\pi^2}{9\varepsilon} + \frac{40\zeta(3)}{3} - \frac{16\pi^2}{9} + \left(\frac{256\pi^2}{9} - \frac{12\pi^4}{5} + \frac{224\zeta(3)}{3}\right)\varepsilon + \left(-\frac{1792\pi^2}{9} + \frac{136\pi^4}{5} - 36\pi^4\log(2) - \frac{2432\zeta(3)}{3} + \frac{5440\pi^2\zeta(3)}{27} + \frac{3730\zeta(5)}{3}\right)\varepsilon^2 + \left(1728 s_6 - \frac{5320\zeta(5)}{3} - \frac{25556\zeta(3)^2}{9} - \frac{2944\pi^2\zeta(3)}{27} + \frac{15872\zeta(3)}{3} - 576\pi^2\mathrm{Li}_4\left(\frac{1}{2}\right) - 24\pi^2\log^4(2) - 84\pi^4\log^2(2) + 144\pi^4\log(2) + \frac{188\pi^6}{27} - \frac{896\pi^4}{5} + \frac{10240\pi^2}{9}\right)\varepsilon^3 + \dots,$$

$$J_6 = G^{(4)}_{1,0,1,1,1,1,1} = -\frac{4\pi^2}{9\varepsilon} - \frac{40\zeta(3)}{3} + \frac{16\pi^2}{9} + \left(-\frac{64\pi^2}{9} + \frac{2\pi^4}{9} + \frac{160\zeta(3)}{3}\right)\varepsilon + \left(\frac{256\pi^2}{9} - \frac{8\pi^4}{9} - \frac{640\zeta(3)}{3} + \frac{1604\pi^2\zeta(3)}{27} - \frac{1120\zeta(5)}{3}\right)\varepsilon^2 + \left(-\frac{1024\pi^2}{9} + \frac{32\pi^4}{9} + \frac{3061\pi^6}{1890} + \frac{2560\zeta(3)}{3} - \frac{6416\pi^2\zeta(3)}{27} + \frac{1400\zeta(3)^2}{9} + \frac{4480\zeta(5)}{3}\right)\varepsilon^3 + \dots,$$

$$J_7 = G^{(5)}_{1,0,1,1,1,1,-1} = \frac{1}{3\varepsilon} - 2\zeta(3) + 4 + \left(\frac{100}{3} - \frac{\pi^2}{18} - \frac{\pi^4}{10} - 12\zeta(3)\right)\varepsilon + \left(240 - \frac{2\pi^2}{3} - \frac{3\pi^4}{5} - \frac{728\zeta(3)}{9} + \frac{\pi^2\zeta(3)}{3} - 52\zeta(5)\right)\varepsilon^2$$

$$+\left(\frac{4816}{3}-\frac{50\pi^2}{9}-\frac{449\pi^4}{120}-\frac{319\pi^6}{1260}-\frac{1616\zeta(3)}{3}+2\pi^2\zeta(3)+\frac{232\zeta(3)^2}{3}-312\zeta(5)\right)\varepsilon^3+\ldots,$$

$$J_8=G^{(5)}_{1,1,1,0,1,1,1}=-\frac{4\pi^2}{9\varepsilon}+\frac{20\zeta(3)}{3}+\frac{16\pi^2}{9}+\left(-\frac{64\pi^2}{9}+\frac{16\pi^4}{9}-\frac{80\zeta(3)}{3}\right)\varepsilon+\left(\frac{256\pi^2}{9}-\frac{64\pi^4}{9}+\frac{320\zeta(3)}{3}+\frac{1514\pi^2\zeta(3)}{27}+\frac{560\zeta(5)}{3}\right)\varepsilon^2$$
$$+\left(-\frac{1024\pi^2}{9}+\frac{256\pi^4}{9}+\frac{6971\pi^6}{1890}-\frac{1280\zeta(3)}{3}-\frac{6056\pi^2\zeta(3)}{27}-\frac{700\zeta(3)^2}{9}-\frac{2240\zeta(5)}{3}\right)\varepsilon^3+\ldots,$$

$$J_9=G^{(6)}_{0,1,1,1,0,1,1}=\frac{1}{3\varepsilon^2}+\left(\frac{8}{3}+\frac{2}{3}\log(2)\right)\frac{1}{\varepsilon}+\frac{2\log^2(2)}{3}+\frac{16\log(2)}{3}+\frac{2\pi^2}{9}+\frac{52}{3}$$
$$+\left(\frac{320}{3}+\frac{16\pi^2}{9}+\frac{104\log(2)}{3}+\frac{4}{9}\pi^2\log(2)+\frac{16\log^2(2)}{3}+\frac{4\log^3(2)}{9}-\frac{140\zeta(3)}{9}\right)\varepsilon$$
$$+\left(\frac{1936}{3}+\frac{104\pi^2}{9}-\frac{179\pi^4}{270}+\frac{640\log(2)}{3}+\frac{32}{9}\pi^2\log(2)+\frac{104\log^2(2)}{3}+\frac{4}{9}\pi^2\log^2(2)+\frac{32\log^3(2)}{9}+\frac{2\log^4(2)}{9}-\frac{1120\zeta(3)}{9}-\frac{280\log(2)\zeta(3)}{9}\right)\varepsilon^2$$
$$+\left(\frac{11648}{3}+\frac{640\pi^2}{9}-\frac{716\pi^4}{135}+\frac{3872\log(2)}{3}+\frac{208}{9}\pi^2\log(2)-\frac{179}{135}\pi^4\log(2)+\frac{640\log^2(2)}{3}+\frac{32}{9}\pi^2\log^2(2)+\frac{208\log^3(2)}{9}+\frac{8}{27}\pi^2\log^3(2)+\frac{16\log^4(2)}{9}+\frac{4\log^5(2)}{45}-\frac{7280\zeta(3)}{9}-\frac{280\pi^2\zeta(3)}{27}-\frac{2240\log(2)\zeta(3)}{9}-\frac{280}{9}\log^2(2)\zeta(3)-\frac{6572\zeta(5)}{15}\right)\varepsilon^3+\ldots,$$

$$J_{10}=G^{(7,-)}_{0,1,1,0,1,1,1}=-\frac{8\pi^2}{9\varepsilon}-\frac{8\zeta(3)}{3}-\frac{16}{9}\pi^2\log(2)-\frac{64\pi^2}{9}+\left(-\frac{512\pi^2}{9}+\frac{40\pi^4}{27}-\frac{128}{9}\pi^2\log(2)-\frac{16}{9}\pi^2\log^2(2)-\frac{64\zeta(3)}{3}-\frac{16\log(2)\zeta(3)}{3}\right)\varepsilon$$
$$+\left(-\frac{4096\pi^2}{9}+\frac{320\pi^4}{27}-\frac{1024}{9}\pi^2\log(2)+\frac{80}{27}\pi^4\log(2)-\frac{128}{9}\pi^2\log^2(2)-\frac{32}{27}\pi^2\log^3(2)-\frac{512\zeta(3)}{3}+\frac{3184\pi^2\zeta(3)}{27}-\frac{128\log(2)\zeta(3)}{3}-\frac{16}{3}\log^2(2)\zeta(3)+24\zeta(5)\right)\varepsilon^2$$
$$+\left(-\frac{32768\pi^2}{9}+\frac{2560\pi^4}{27}+\frac{52\pi^6}{7}-\frac{8192}{9}\pi^2\log(2)+\frac{640}{27}\pi^4\log(2)-\frac{1024}{9}\pi^2\log^2(2)+\frac{80}{27}\pi^4\log^2(2)-\frac{256}{27}\pi^2\log^3(2)-\frac{16}{27}\pi^2\log^4(2)-\frac{4096\zeta(3)}{3}+\frac{25472\pi^2\zeta(3)}{27}-\frac{1024\log(2)\zeta(3)}{3}+\frac{6368}{27}\pi^2\log(2)\zeta(3)-\frac{128}{3}\log^2(2)\zeta(3)-\frac{32}{9}\log^3(2)\zeta(3)+\frac{2032\zeta(3)^2}{9}+192\zeta(5)+48\log(2)\zeta(5)\right)\varepsilon^3+\ldots,$$

$$J_{11}=G^{(7,+)}_{0,0,1,1,1,2,0}=\frac{1}{3\varepsilon^2}+\left(\frac{4}{3}+\frac{2\log(2)}{3}\right)\frac{1}{\varepsilon}+\frac{2\log^2(2)}{3}+\frac{8\log(2)}{3}-\frac{2\pi^2}{9}+\frac{28}{3}+\left(\frac{160}{3}-\frac{8\pi^2}{9}+\frac{56\log(2)}{3}-\frac{4}{9}\pi^2\log(2)+\frac{8\log^2(2)}{3}+\frac{4\log^3(2)}{9}-\frac{188\zeta(3)}{9}\right)\varepsilon$$
$$+\left(\frac{976}{3}-\frac{56\pi^2}{9}-\frac{9\pi^4}{10}+\frac{320\log(2)}{3}-\frac{16}{9}\pi^2\log(2)+\frac{56\log^2(2)}{3}-\frac{4}{9}\pi^2\log^2(2)+\frac{16\log^3(2)}{9}+\frac{2\log^4(2)}{9}-\frac{752\zeta(3)}{9}\right.$$

$$
-\frac{376\log(2)\zeta(3)}{9}\bigg)\varepsilon^2+\bigg(\frac{5824}{3}-\frac{320\pi^2}{9}-\frac{18\pi^4}{5}
+\frac{1952\log(2)}{3}-\frac{112}{9}\pi^2\log(2)-\frac{9}{5}\pi^4\log(2)
+\frac{320\log^2(2)}{3}-\frac{16}{9}\pi^2\log^2(2)+\frac{112\log^3(2)}{9}
-\frac{8}{27}\pi^2\log^3(2)+\frac{8\log^4(2)}{9}+\frac{4\log^5(2)}{45}
-\frac{5264\zeta(3)}{9}+\frac{376\pi^2\zeta(3)}{27}-\frac{1504\log(2)\zeta(3)}{9}
-\frac{376}{9}\log^2(2)\zeta(3)-\frac{7532\zeta(5)}{15}\bigg)\varepsilon^3+\bigg(\frac{35008}{3}
-\frac{1952\pi^2}{9}-\frac{126\pi^4}{5}-\frac{5617\pi^6}{2835}+\frac{11648\log(2)}{3}
-\frac{640}{9}\pi^2\log(2)-\frac{36}{5}\pi^4\log(2)+\frac{1952\log^2(2)}{3}
-\frac{112}{9}\pi^2\log^2(2)-\frac{9}{5}\pi^4\log^2(2)+\frac{640\log^3(2)}{9}
-\frac{32}{27}\pi^2\log^3(2)+\frac{56\log^4(2)}{9}-\frac{4}{27}\pi^2\log^4(2)
+\frac{16\log^5(2)}{45}+\frac{4\log^6(2)}{135}-\frac{30080\zeta(3)}{9}
+\frac{1504\pi^2\zeta(3)}{27}-\frac{10528\log(2)\zeta(3)}{9}
+\frac{752}{27}\pi^2\log(2)\zeta(3)-\frac{1504}{9}\log^2(2)\zeta(3)
-\frac{752}{27}\log^3(2)\zeta(3)+\frac{17672\zeta(3)^2}{27}-\frac{30128\zeta(5)}{15}
-\frac{15064\log(2)\zeta(5)}{15}\bigg)\varepsilon^4+\dots,
$$

$$
J_{12}=G^{(7,+)}_{0,1,1,0,1,1,1}=+\frac{4\pi^2}{9\varepsilon}-\frac{8\zeta(3)}{3}+\frac{8}{9}\pi^2\log(2)
+\frac{32\pi^2}{9}+\bigg(\frac{256\pi^2}{9}-\frac{20\pi^4}{27}+\frac{64}{9}\pi^2\log(2)
+\frac{8}{9}\pi^2\log^2(2)-\frac{64\zeta(3)}{3}-\frac{16\log(2)\zeta(3)}{3}\bigg)\varepsilon
+\bigg(\frac{2048\pi^2}{9}-\frac{160\pi^4}{27}+\frac{512}{9}\pi^2\log(2)
-\frac{40}{27}\pi^4\log(2)+\frac{64}{9}\pi^2\log^2(2)+\frac{16}{27}\pi^2\log^3(2)
-\frac{512\zeta(3)}{3}-\frac{1520\pi^2\zeta(3)}{27}-\frac{128\log(2)\zeta(3)}{3}
-\frac{16}{3}\log^2(2)\zeta(3)+24\zeta(5)\bigg)\varepsilon^2+\bigg(\frac{16384\pi^2}{9}
-\frac{1280\pi^4}{27}-\frac{218\pi^6}{63}+\frac{4096}{9}\pi^2\log(2)
-\frac{320}{27}\pi^4\log(2)+\frac{512}{9}\pi^2\log^2(2)-\frac{40}{27}\pi^4\log^2(2)
+\frac{128}{27}\pi^2\log^3(2)+\frac{8}{27}\pi^2\log^4(2)-\frac{4096\zeta(3)}{3}
-\frac{12160\pi^2\zeta(3)}{27}-\frac{1024\log(2)\zeta(3)}{3}
-\frac{3040}{27}\pi^2\log(2)\zeta(3)-\frac{128}{3}\log^2(2)\zeta(3)
-\frac{32}{9}\log^3(2)\zeta(3)+\frac{2032\zeta(3)^2}{9}+192\zeta(5)
+48\log(2)\zeta(5)\big)\varepsilon^3+\dots,
$$

$$
J_{13}=G^{(7,+)}_{1,1,1,1,1,1,1}=-\frac{8\pi^2}{9\varepsilon}+\frac{40\zeta(3)}{3}
-\frac{16}{9}\pi^2\log(2)-\frac{16\pi^2}{9}+\bigg(\frac{256\pi^2}{9}-\frac{292\pi^4}{135}
-\frac{32}{9}\pi^2\log(2)-\frac{16}{9}\pi^2\log^2(2)+\frac{224\zeta(3)}{3}
+\frac{80\log(2)\zeta(3)}{3}\bigg)\varepsilon+\bigg(-\frac{1792\pi^2}{9}+\frac{3904\pi^4}{135}
+\frac{512}{9}\pi^2\log(2)-\frac{4904}{135}\pi^4\log(2)-\frac{32}{9}\pi^2\log^2(2)
-\frac{32}{27}\pi^2\log^3(2)-\frac{2432\zeta(3)}{3}+\frac{4912\pi^2\zeta(3)}{27}
+\frac{448\log(2)\zeta(3)}{3}+\frac{80}{3}\log^2(2)\zeta(3)+1176\zeta(5)\bigg)\varepsilon^2
+\bigg(\frac{10240\pi^2}{9}-\frac{5312\pi^4}{27}+\frac{11908\pi^6}{2835}
-\frac{3584}{9}\pi^2\log(2)+\frac{25088}{135}\pi^4\log(2)
+\frac{512}{9}\pi^2\log^2(2)-\frac{17864}{135}\pi^4\log^2(2)
-\frac{64}{27}\pi^2\log^3(2)-\frac{16}{27}\pi^2\log^4(2)+\frac{15872\zeta(3)}{3}
+\frac{896\pi^2\zeta(3)}{27}-\frac{4864\log(2)\zeta(3)}{3}
+\frac{9824}{27}\pi^2\log(2)\zeta(3)+\frac{448}{3}\log^2(2)\zeta(3)
+\frac{160}{9}\log^3(2)\zeta(3)-\frac{16496\zeta(3)^2}{9}-864\zeta(5)
+2352\log(2)\zeta(5)\big)\varepsilon^3+\dots,
$$

$$
\begin{aligned}
J_{14} = G^{(7,-)}_{1,1,1,1,1,1,1} &= \frac{16\pi^2}{9\varepsilon} + \frac{40\zeta(3)}{3} \\
&+ \frac{32}{9}\pi^2\log(2) + \frac{32\pi^2}{9} + \Big(-\frac{512\pi^2}{9} + \frac{548\pi^4}{135} \\
&+ \frac{64}{9}\pi^2\log(2) + \frac{32}{9}\pi^2\log^2(2) + \frac{224\zeta(3)}{3} \\
&+ \frac{80\log(2)\zeta(3)}{3}\Big)\varepsilon + \Big(\frac{3584\pi^2}{9} - \frac{4496\pi^4}{135} \\
&- \frac{1024}{9}\pi^2\log(2) + \frac{5416}{135}\pi^4\log(2) + \frac{64}{9}\pi^2\log^2(2) \\
&+ \frac{64}{27}\pi^2\log^3(2) - \frac{2432\zeta(3)}{3} - \frac{10544\pi^2\zeta(3)}{27} \\
&+ \frac{448\log(2)\zeta(3)}{3} + \frac{80}{3}\log^2(2)\zeta(3) + 1176\zeta(5)\Big)\varepsilon^2 \\
&+ \Big(-\frac{20480\pi^2}{9} + \frac{5440\pi^4}{27} - \frac{4472\pi^6}{2835} \\
&+ \frac{7168}{9}\pi^2\log(2) - \frac{26272}{135}\pi^4\log(2) \\
&- \frac{1024}{9}\pi^2\log^2(2) + \frac{18376}{135}\pi^4\log^2(2) \\
&+ \frac{128}{27}\pi^2\log^3(2) + \frac{32}{27}\pi^2\log^4(2) + \frac{15872\zeta(3)}{3} \\
&- \frac{5824\pi^2\zeta(3)}{27} - \frac{4864\log(2)\zeta(3)}{3} \\
&- \frac{21088}{27}\pi^2\log(2)\zeta(3) + \frac{448}{3}\log^2(2)\zeta(3) \\
&+ \frac{160}{9}\log^3(2)\zeta(3) - \frac{16496\zeta(3)^2}{9} - 864\zeta(5) \\
&+ 2352\log(2)\zeta(5)\Big)\varepsilon^3 + \ldots .
\end{aligned}
$$

In the above results the factor $(i\pi^{d/2}e^{-\gamma_E\varepsilon})^3$ is implied on the right-hand side. Furthermore, $s_6 = \zeta(\{-5,-1\},\infty) + \zeta(6) = \sum_{i=1}^{\infty}\frac{(-1)^i}{i^5}\sum_{j=1}^{i}\frac{(-1)^j}{j}$, and we show the ε expansion terms up to the order which is needed for the static potential. The results for $J_1,\ldots,J_{14}$ have been checked numerically with the help of the program `FIESTA` [12,13].

4. Result for a_3

For completeness we repeat the result for the non-fermionic part of a_3, $a_3^{(0)}$, which has been obtained in Refs. [8,9]

$$
a_3^{(0)} \;=\; 502.24(1)\, C_A^3 - 136.39(12)\, \frac{d_F^{abcd}d_A^{abcd}}{N_A}\,,
$$

where $C_A = N_c$ and $d_F^{abcd}d_A^{abcd}/N_A = (N_c^3 + 6N_c)/48$ in the case of $SU(N_c)$. Since for three master integrals the highest ε expansion coefficient is only known numerically (see Ref. [10]) no analytical result is available yet for $a_3^{(0)}$. Note, however, that the achieved accuracy is sufficient for all foreseeable applications.

Acknowledgements

This work is supported by DFG through project SFB/TR 9 “Computational Particle Physics” and RFBR, grant 08-02-01451. V.S. appreciates the financial support (for the participation in the workshop) of the Institute of Theoretical Particle Physics of KIT.

REFERENCES

1. A. V. Smirnov, V. A. Smirnov and M. Steinhauser, Phys. Lett. B **668** (2008) 293 [arXiv:0809.1927 [hep-ph]].
2. W. Fischler, Nucl. Phys. B **129** (1977) 157.
3. A. Billoire, Phys. Lett. B **92** (1980) 343.
4. M. Peter, Phys. Rev. Lett. **78** (1997) 602 [arXiv:hep-ph/9610209].
5. M. Peter, Nucl. Phys. B **501** (1997) 471 [arXiv:hep-ph/9702245].
6. Y. Schroder, Phys. Lett. B **447** (1999) 321 [arXiv:hep-ph/9812205].
7. B. A. Kniehl, A. A. Penin, M. Steinhauser and V. A. Smirnov, Phys. Rev. D **65** (2002) 091503 [arXiv:hep-ph/0106135].
8. A. V. Smirnov, V. A. Smirnov and M. Steinhauser, Phys. Rev. Lett. **104** (2010) 112002 [arXiv:0911.4742 [hep-ph]].
9. C. Anzai, Y. Kiyo and Y. Sumino, Phys. Rev. Lett. **104** (2010) 112003 [arXiv:0911.4335 [hep-ph]].
10. A. V. Smirnov, V. A. Smirnov and M. Steinhauser, arXiv:1001.2668 [hep-ph].
11. A. V. Smirnov, JHEP **0810** (2008) 107 [arXiv:0807.3243 [hep-ph]].
12. A. V. Smirnov and M. N. Tentyukov, Comput. Phys. Commun. **180** (2009) 735 [arXiv:0807.4129 [hep-ph]].
13. A. V. Smirnov, V. A. Smirnov and M. Tentyukov, arXiv:0912.0158 [hep-ph].